On Subtractive Operations, Subtractive Numbers, and Purportedly Negative Numbers in Old Babylonian Mathematics

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in memoriam

The uses of the terms and expressions *nasāḫum, eli . . . watārum, ḫarāṣum, tabālum, šutbûm* and LÁ (LAL) in Old Babylonian mathematical texts are investigated. The two first operations turn out to be genuine mathematical terms, designating concrete removal and comparison, respectively; *ḫarāṣum, tabālum* and *šutbûm* are terms from everyday life used to formulate "dressed problems" and hence also occasionally more or less metaphorically for subtraction by removal within the description of procedures. LÁ (on one occasion *maṭûm*) is used as a substitute for *eli . . . watārum* when stylistic or similar reasons require that the smaller of two magnitudes to be compared be mentioned first.

The claim sometimes made (going back to misreadings of Neugebauer) that LÁ has to do with a Babylonian concept of negative numbers is thus unfounded.

In a number of earlier studies, some of them as yet unpublished,[1] I have investigated the panoply of operations applied in Old Babylonian so-called "algebra". Among the results is a distinction between two main "additive operations", *waṣābum* (with logogram DAḪ) and *kamārum* (logograms GAR.GAR and UL.GAR), which have quite distinct roles within the texts, and a halfway corresponding distinction between two main "subtractions", *nasāḫum* (ZI) and *eli . . . watārum* (UGU . . . DIRIG). Without pursuing the matter I have also taken note of the apparently distinct use of other subtractive operations (*ḫarāṣum, maṭûm*) and of what looks as evidence for a category of "subtractive [role of a] number".

The present paper represents an attempt to pursue these latter questions systematically, and to connect them with a claim which is occasionally made – *viz* that Old Babylonian calculators had a concept of negative number.

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[1] Among the published items, I shall only refer to Høyrup 1990, which contains the most thorough presentation and discussion of evidence and results.

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As a mathematical term, *waṣābum* designates a concrete process where an entity A is joined or "appended" to another entity C of the same kind – cf. the etymology of Latin *ad-do*. In the process, C conserves its identity, and A is absorbed. For this reason, the sum by this process possesses no particular name of its own. A convenient model for this kind of additive thinking is suggested by the derivative *ṣibtum*, "interest": If interest is added to *my* bank-account the increased balance remains *my account*.

kamārum is the addition where the single contributions are brought together or "accumulated" into a common heap – cf. the etymology of Latin *ac-cumulo*. In this process, the single contributions loose their identity, and the heap (i.e., the sum) therefore has a particular name, the *kimirtum* (the text AO 8862 employs the plural *kimrātum*, referring to the composite nature of the heap).

Those second- and third-degree problems which add entities of different dimension (be it length and area, as, e.g., in BM 13901, or volume and area, as in BM 85200 + VAT 6599), normally do this by "accumulation" (I shall discuss one characteristic exception below). The implication appears to be that this is, or can at least be, a real (i.e., an arithmetical) addition of measuring numbers. "Appending", on the other hand, is additive but not arithmetical, putting together concrete entities; phrases like

30 . . . *a-na* 29,30 *tu-ṣa-ab-ma* 30 *mi-it-ḫar-tum*
30 . . . to 29,30 you append: 30,0 the side of the square

(BM 13901, obv. i 8) should be read as descriptions of a concrete procedure where the concomitant arithmetical operations are *implied*.

nasāḫum, "to tear out", is the reversal of appending. This is made evident, among other things, by numerous texts where they occur in parallel. As pointed out by Vajman (1961: 100), addition and subtraction (by these two operations) of the semi-difference $d = {}^{a-b}/_2$ between two magnitudes a and b to and from their semi-sum $r = {}^{a+b}/_2$ are normally organized in such a way that d is first torn out from one copy of r and next appended to another copy, i.e., *the same* piece d is simply transferred from one to the other.[2]

[2] The geometrical interpretation of the algebra texts makes this transfer even more meaningful than it was to Vajman – cf. Høyrup 1990: 264.
Vajman's rule is not without exceptions. See, e.g., BM 13901 Nos 8, 9 and 12.

All the more strange is the occurrence of *nasāḫum* in the beginning of certain algebra problems as the counterpart of *kamārum*. In order to see what goes on we may look at the statements of the first problems of the mathematical procedure text BM 13901:

No. 1

1. A.ŠÀ^{lam} *ù mi-it-ḫar-ti ak-m[ur-m]a* 45-E 1 *wa-ṣi-tam*
The surface and my confrontation I have accumulated: 45′ is it. 1, the projection,
2. *ta-ša-ka-an* . . .
you pose. . . .
4. . . . 30 *mi-it-ḫar-tum*
. . . 30′ the confrontation.

No. 2

5. *mi-it-ḫar-ti lìb-bi* A.ŠÀ *[a]s-sú-uḫ-ma* 14,30-E 1 *wa-ṣi-tam*
My confrontation inside the surface I have torn out: 14, 30 is it. 1, the projection,
6. *ta-ša-ka-an* . . .
you pose. . . .
8. . . . 30 *mi-it-ḫar-tum*
. . . 30′ the confrontation.

No. 3

9. *ša-lu-uš-ti* A.ŠÀ *as-sú⟨-uḫ-ma⟩ ša-lu-uš-ti mi-it-ḫar-tim a-na lìb-bi*
The third of the surface I have torn out. The third of the confrontation to the inside
10. A.ŠÀ^{lim} *ú-ṣi-ib-ma* 20-E . . .
of the surface I have appended: 20′ is it. . . .
15. . . . [30] *mi-it-ḫar-tum*
. . . [30′], the confrontation.

No. 4

16. *ša-lu-uš[-ti* A.ŠÀ *as-sú-uḫ-ma* A.ŠÀ *ù m]i-i[t-ḫa]r-ti*
The third of the surface I have torn out: The surface and my confrontation
17. *ak-mur-ma* ⌜4,46,40-E . . .]
I have accumulated, $4'46^{0}40'$ is it. . . .

23. 20 [*mi-i*]*t-ḫar-tum*
20, the confrontation

First a few words to the translation:

Sexagesimal place value numbers are translated according to Thureau-Dangin's system, `, " etc. indicating increasing and ′, ″ etc. decreasing orders of magnitude.

"Confrontation" stands for *mitḫartum* and is meant to render the connection between the latter word and *maḫārum*. What should be thought of is a quadratic configuration consisting of four equal lines confronting each other; numerically, the "confrontation" is determined by the length of one side (in other words, the "confrontation"/*mitḫartum* can be imagined as the side of a square which presupposes and implies the presence of the square as inseparably as, to our thinking, a quadratic area presupposes and implies the presence of a quadratic perimeter). The "surface" (A.ŠÀ) is the area of the quadratic figure. (The word "surface" is used instead of "area" to translate A.ŠÀ in order to emphasize that the primary meaning of the term is the geometrical extension – "a field" – and that the number measuring the area of this extension is only a secondary meaning).

The "projection" translates *waṣītum*, and should be understood as a breadth *1* which, when given to a line (in case a "confrontation") of length L transforms it into a rectangle of area $1 \cdot L = L$.

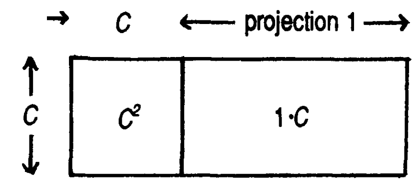
"Inside" is meant to render the use of *libbum* in our text, where it seems to serve as nothing more than an indication that the entity to which something is appended or from which something is torn out possesses bulk or body.

With this in mind we may start by looking at *problem No 1.* At first we are told that the accumulation of [the measuring numbers of] area and side of a square configuration is 45′. In order to make geometrical sense of this, the "projection" 1 is "posed" as in *Figure 1.* It is not said explicitly, but in this way the rectangle $1 \cdot C$ can be "appended" to the surface $C \cdot C$. Cut-and-paste manipulation of the resulting figure (whose area is known to be 45′) allows a final disentanglement of the confrontation.

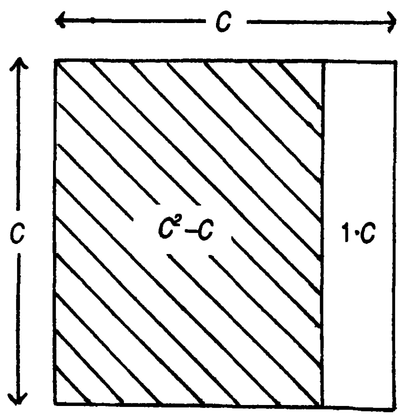
Problem No 2 is similar, but subtracts the side from the area by tearing out the confrontation from the surface. It is only as a first step in the procedure that the projection is posed explicitly, but already in the statement is it implicitly presupposed by the use of the verb *nasāḫum* – cf. *Figure 2,* where the shaded area shows what remains when the confrontation has been torn out.

From *No 2* alone, it is true, we cannot be sure that a "projection" is implicitly presupposed. After all, like *kamārum, nasāḫum* might operate on the measuring numbers. *Nos 3–4,* however, shows us that this is not the case. The statement of *No 3* starts by tearing out a third of the area, before it adds a third of the confrontation. This time, however, the confrontation (actually its third) is *appended,* see *Figure 3.*

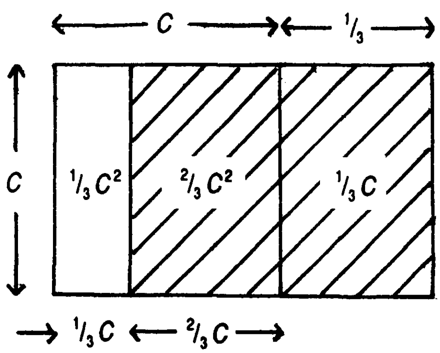
In *No 4*, on the other hand, where a third of the surface is again torn out, the addition of a confrontation is another accumulation. In between, however, an intermediate step has been inserted, *viz* a reference to the reduced surface as a surface of its own. (*Figure 4*).



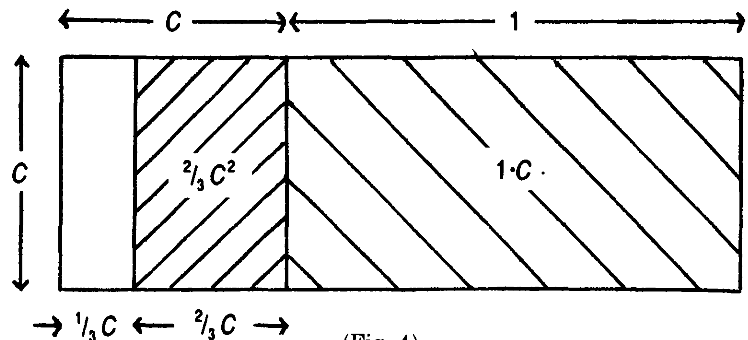
(Fig. 1)



(Fig. 2)



(Fig. 3)



(Fig. 4)

Together, the two formulations suggest the following interpretation: Tearing-out a part of the surface transfers the process to the level of concrete geometric manipulations; once we are there, the only additive operation at our disposal is appending, which presupposes that the confrontation to be added is implicitly provided with a "projection". This is what happens in *No 3*. *No 4*, on its part, by speaking of the result of the tearing as a "surface", takes note of it as an entity of its own, possessing its own measuring number. *This number*, and not the palpable surface resulting from the tearing, can be accumulated with the [measure of the] confrontation, as it happens in *No 4*.

The present interpretation of *No 3* presupposes that the Babylonian calculators were predisposed to think of a line segment as provided automatically with a standard width (a "projection") *1* – an idea which is rather unfamiliar to our post-Euclidean mode of geometrical thought. We are equally unprepared, however, to think of a surface as provided with a standard height. The latter idea, as it is well known, was the very foundation of Babylonian volume metrology, which did not distinguish the area measure sar (NINDAN^2) from the volume measure sar (NINDAN^2 · KÙŠ) – meaning that a volume was measured by the area it would cover if distributed with the standard height 1 cubit[3].

Mathematical texts also tell us that lines were understood as representing the rectangles of which they were the sides – thus, e.g., the confrontation represented the whole quadratic configuration. There is thus nothing strange in the shift from metro-numerical to concrete representation in *No 3*.

This brings us back to *No 2*: if tearing-out in *No 3* enforces a shift to concrete representation in *No 3*, it cannot be the reverse of an accumulation; even when the confrontation is torn out in *No 2* it must thus be provided with a tacit "projection *1*".

The expression *eli . . . watārum*/UGU . . . DIRIG was introduced above as the other main subtractive operation. "*P* UGU *Q R* DIRIG" can be translated *de verbo ad verbum* as "*P* over *Q R* goes beyond" or, if this principle is abandoned, "*P* exceeds *Q* by *R*". Arithmetically, this means that $P–Q = R$. This "subtraction by comparison"

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[3] A less well-known but even more pertinent parallel is provided by the names of the area units NINDAN and EŠE. As pointed out by Powell (1972: 185), "for the definition of both the NÍG and the eše, a rectangle with a fixed side of 1,0 NÍG is assumed as constant. If the base of the rectangle is one NÍG in length, the plot is termed a NÍG; if the base is one eše in length, it is termed an eše".

It may be of interest that Egyptian area metrology refers to similar conceptions: a "cubit of land" is a strip of standard length 100 cubit and one cubit wide; a "thousand of land" equals one thousand such strips. See Peet 1923: 25.

is used in BM 13901 in cases where one confrontation is told to exceed another by a certain amount or fraction: *mitḫartum* UGU *mitḫartim* 10 *itter* (obv. ii 4, rev. i 40) or *mitḫartum* UGU *mitḫartim sebiātim itter* (obv. ii 20). The outcome of the operation can also be used for further operations even if unknown, in which case it is spoken of as *mala mitḫartum* UGU *mitḫartim itteru*, "so much as the confrontation over the confrontation goes beyond", or simply, if the identity of *P* and *Q* goes by itself, as DIRIG, "the excess". In all cases, as we see, entities of the same kind are compared. In contrast to the remainder after a tearing-out, the excess does not take over the identity of *P*.

The relation between accumulating, appending, tearing-out, exceeding and "reclaiming" (another quasi-subtractive operation or term) is highlighted by a problem collection from Susa.[4] The first sequence of problems tells the side of a square – the *confrontation* – and asks for a varying multiple of the *length* (UŠ – numerically the same as the confrontation).[5] The next sequence (*section 3*, an intermediate sequence being broken off) accumulates the confrontation and a multiple of the length, while *section 4* tells how much the confrontation exceeds a multiple of the length. *Section 5* gives the confrontation and asks for a multiple of the surface, while *section 6* gives a multiple of the surface and asks for the confrontation. *Section 7* tells the confrontation and asks for the area of [the square built on] a multiple of the length, while *section 8* accumulates the area and the area of a multiple of the length, and *section 9* tells the excess of the area over the area of a (sub)multiple of the length.

Section 10 introduces mixed second-degree problems, *appending* a multiple of one length to the "surface of the confrontation". *Section 11*, the subtractive counterpart, falls into two subsections. In the first, a multiple of the length is torn out from the surface, leaving a known remainder; in the second, where the multiple of the length is larger than or equal to the area, the multiple of the

[4] TMS V. Bruins' transcriptions, translations and commentaries in the edition of the mathematical Susa texts abound in mistakes, and even if text V has been treated with more attention than most one should still base the discussion upon the transliteration and the autography.

[5] I disregard this other aspect of the text – the principles according to which the multiples are systematically varied and designated – having dealt with the question elsewhere (Høyrup 1990a: 303–305).

length is told to exceed the surface by so and so much, or to be "as much as the surface" (*kīma* A.ŠÀ). *Section 12,* finally, states that a certain part of the surface has been "withdrawn" (*tabālum*) and tells the remainder, asking for the confrontation,[6] after which follow problems of a different type, to which we shall return.

The text tells neither procedures nor solutions, but these can be easily reconstructed from other mathematical texts. It turns out that "accumulation" is used where further operations will take place on the arithmetical level, e.g., by an argument of the type "single false position" (*sections 3* and *8*); in both cases, the entities to be added are of the same kind (a precondition, indeed, for the application of purely arithmetical techniques). The corresponding subtractive sequences (*sections 4* and *9*) go by comparison, corroborating the observation made on BM 13901 that only entities of the same kind can be compared.

Cases where the physical outcome of an additive procedure and not its arithmetical expression will be the basis for further operations (i.e., the additions of area and sides in *section 10*) are made by *appending.* Our present text thus presupposes that sides are already provided with an implicit "projection", carrying hence a surface with them. The corresponding subtractions of *section 11a* are made by *tearing-out.*[7]

This way to subtract sides from the area corresponds to what happens in BM 13901. *Section 11b*, on the other hand, is unusual.

It is unusual already for its mathematical content. No other mixed second-degree problems with a single unknown of this structure are known (the type in question, ax-$x^2 = b$, is the one which possesses two positive roots), even though several complex

[6] The phrase is, in the first example of this section, "$^1/_3$ A.ŠÀ *it-ba-al* ÍB.SI A.ŠÀ 10 LAGAB *mi-nu*". LAGAB is the standard logogram used in the text for *mitḫartum,* for which ÍB.SI$_8$ (regularly written ÍB.SI in the Susa tablets) is used in a number of other texts, and which in any case has a normal use quite close to that of *mitḫartum:* The standard phrase "*A*-E *r* ÍB.SI$_8$", often translated "r is the square-root of *A*", should rather be "*A* makes *r* equilateral", i.e., when formed as a square, the area *A* will produce the side *r.*

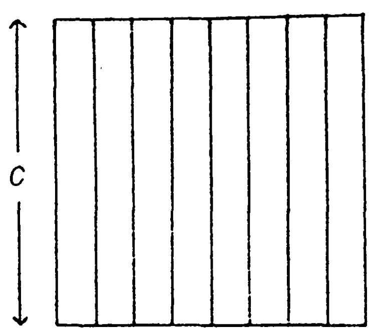
The side of the square is 30 and the corresponding area hence 15′; the most plausible reading of the phrase therefore appears to be "$^1/_3$ of the surface (somebody) has withdrawn (regarding) the surface of the equilateral, 10′. What (is) the confrontation?".

[7] TMS VI contains parallels to sections 10 and 11a from TMS V. Here, similarly, sides are appended to and torn out from the area.

problems are solved in a way which suggests that they were reduced to this type.[8]

But even the formulation is unexpected. What is the reason that sides (provided tacitly with a "projection") can be torn out from an area, but an area apparently not from a multiple of sides? In the absence of parallel texts only a tentative explanation can be given.

We may start from the general semantic of the term *nasāḫum*. In agreement with the translation, one can only "tear out" what is already part of the total: if B is not a part of A one can at best tear out "so much as B" (*mala B*) from A. That n sides of the square on C can be torn out thus appears to imply that the square really consists of C strips each C long and *1* wide – evidently an idea which is close at hand once the side is thought of as a similar strip, cf. *Figure 5*. Since the area cannot be torn out from a multiple $n \cdot C$ of the side ($n > C$), however, it seems that the reverse process (converting an adequate number of strips into a quadratic figure) does not take place automatically. Instead, it is told *by how much it is impossible* to tear out the n sides from the surface, and in one case that tearing-out in the proper sense cannot be performed because everything will be removed.



(Fig. 5)

[8] Thus IM 52301 No 2 (see Høyrup 1990: 341) and BM 85196, rev. ii 7–21 (see Høyrup 1985: 58).

The corresponding problem with two unknowns, $xy = b$, $x + y = a$, is of course well-known. There is thus no doubt that the problem would have to be solved by means of the same geometric cut-and-paste technique as the other mixed second-degree problems of the tablet.

Section 12 confronts us with another puzzle: Why doesn't it make use of the same operation as *section 9*, since the method by which its problems are solved will have been more or less the same? Once again, the absence of parallels in other mathematical texts prevents us from knowing with certainty. However, this absence also suggests the beginning of an answer. A term which is so rare but which is none the less used in a mathematically quite trivial context can hardly represent a genuine mathematical operation. It will rather be an "everyday" term, i.e., a term taken from the extra-mathematical facets of scribal life.

Now, *tabālum*, "to withdraw", is indeed used routinely in connection with fields (the extra-mathematical meaning of A.ŠÀ), in particular when a whole field or a specified part of it is reclaimed from the owner by legal action.[9] Since the term turns up in *section 12*, after the apex of mathematical sophistication represented by second-degree algebra, and since the following part of the tablet deals with squares inscribed concentrically into squares, a subject derived somehow from geometrical practice, the problems about areas with withdrawn parts may plausibly constitute a first section of "dressed problems". The authentic meaning of its first problem (cf. *note 6*) will then be something like this: "from a quadratic field, somebody has reclaimed $1/3$ of the area, and what is left amounts to 10′ sar. What is the side of the square?".[10]

This interpretation explains another enigmatic feature of the text. In ordinary mathematical texts, the statement is made by the teacher in the first person singular preterit, "I have done so and so". This is so much a routine that Bruins overlooked the third person used in the text, translating *it-ba-al* as "j'ai ôté . . .". Only if the problems do not deal with a configuration constructed or

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9 An Old Babylonian example is found in Walters 1970: 64 (text 45 lines 15 and 18, cf. the note to line 15); a Kassite example specifying that "reduction" (*niširtum*) of the field is involved is in Scheil 1905: 36 (iv 15f.).

10 The problems of section 12 will thus provide nice examples of what Karen Nemet-Nejat (1993) has spoken of as "mathematical texts as a reflection of everyday life in Mesopotamia".

My choice of order of magnitude has been made so as to fit real-life fields as closely as possible – from the text alone, the area left over might as well be 10′ sar, in which case the square field would be approcimately 3 m by 3 m. Evidently, this choice is arbitrary – neither Babylonian nor modern mathematics teachers care much whether their numerical data are plausible.

prepared by the teacher but with a fictional juridical case, the subject of the action should really be a *somebody*, a third person.

A shift like this within the same tablet, from "pure" calculation to practical computation (real or fictional) is not unprecedented in the Old Babylonian mathematical corpus. One example can be found in BM 13901, where the penultimate problem (No 23) is probably a surveyors' "recreational" puzzle (cf. Høyrup 1990: 275, 352). Other instances are IM 52301, where an excerpt from an IGI.GUB table follows upon two second-degree "algebra" problems, and AO 8862, where second-degree algebra problems are followed by (still artificial) problems on brick-carrying, involving both practical metrology and a "house builder" (*itinnum*). Most relevant of all texts is perhaps IM 52916 (tablet "1" of the "Tell Ḫarmal compendium"), like TMS V a list of problems without solution (actually only *problem types*, since even the given numbers are not stated), which starts out with long sequences of second-degree problems, appending sides to or tearing them out from the area, continues with IGI.GUB-factors for geometric figures and with inscription of geometric figures into other figures, and closes with work norms and other practical computational topics.

In the present text, *tabālum* is thus after all probably not to be read as a mathematical technical term, and still less as the name for a distinct mathematical operation. *nasāḫum* and *eli* ... *watārum*, on the other hand, which are used not only in statements but also in the description of mathematical procedures, are technical terms for genuine mathematical operations, or at least as technical as any term in Babylonian mathematics. In two other texts, *tabālum* seems to get closer this role. In YBC 4608, obv. 24 and 27, a line d is "withdrawn" from an entity which is known already to represent the sum $d+b$ of two opposing sides of a quadrangle; the reason for the choice of this specific term may thus be the no less specific situation that d is precisely what can "justly" be withdrawn. In YBC 4662, obv. 9, however, the term is used during the solution of an ordinary second-degree problem in a place where *nasāḫum* would be the standard choice, – and is indeed the actual choice of the parallel passages rev. 9 of the companion text YBC 4663.

tabālum is not the only term which moves imperceptibly between extra-mathematical and mathematical discourse without ever achieving the status of a genuine mathematical term. Similar cases are offered by the verbs *ḫarāṣum*, "to cut off", and *šutbûm* (*tebûm* III), "to make leave", "to remove".

The occurrences of the former in BM 85196 No 18, rev. ii 19,23f. are obvious references to non-mathematical parlance. They simply refer to the cutting-off of parts of silver coils (ḪAR) used for payment. The way the term appears in the fragment VAT 6546 may be inspired from this meaning, since something is cut off from a profit (*nēmelum*); so much is clear, on the other hand, that the term occurs (twice) *inside* the description of the procedure, i.e., that it describes computational steps. This is also the case in AO 6770, No 3. This problem deals with a stone, from which something has been removed and to which something has been added; but *ḫarāṣum* turns up inside the procedure, while the verb of the statement is ZI, "to tear out".

AO 8862, on the other hand, employs the term to describe an indubitable mathematical operation along with *nasāḫum*; so do the twin texts YBC 4663 and 4662. As a general rule, *ḫarāṣum* is used in these three tablets when something is removed from a linear entity; alternatively, *nasāḫum* without *libbi* may be used. In cases where a piece of surface is removed from another surface, the expression used is *ina libbi nasāḫum* (or, as mentioned above, *ina libbi tabālum*).[11]

Only two passages of AO 8862 do not agree with these rules. In iii 11–12, *nasāḫum* without *libbi* is used when a piece of surface is torn out from another. Instead, however, the subtrahend is explicitly told to be a surface (A.ŠÀ). In ii 10–11, finally, *ḫarāṣum* is used even though surfaces (length and width provided with a projection) seem to be involved (see Høyrup 1990: 317).

There is thus no absolute distinction between the two operations. There is, however, an outspoken tendency to keep in mind the concrete character of the process which goes on and to make this visible through the imagery which is inherent in the description – through a distinction between "cutting" and "tearing", between use and non-use of *libbi*, or by an explicit epithet A.ŠÀ.

The merely relative character of the distinction between *nasāḫum* and *ḫarāṣum* in confirmed by a final text where *ḫarāṣum* occurs: in YBC 4675 (and its partial doublet YBC 9852), this term is used for both surfaces and lines; *nasāḫum*, on the other hand, is totally absent.

[11] It should be observed that this use of *libbi* to distinguish between removal from surfaces and from linear entities does not hold outside the small text group in question. BṂ 13901, e.g., uses *libbi* (and *libba*) indiscriminately in both cases.

The case of *šutbûm* is somewhat simpler. In extra-mathematical contexts, it is often used when you remove something or make somebody leave that *should* in fact be removed or go away: making workers go out for work; removing guilt, demons, garbage; taking a statue from its pedestal for use in a procession; etc. (see AHw. 1343). Its use as a quasi-mathematical term may derive from the same idea. In one case, the original magnitude A of a measuring reed is to be found when its length after loss of one 5th is 20′ NINDAN: 5 is inscribed (*lapātum*), so to speak as a model of the original reed; 1 (i.e., one fifth) is removed, leaving 4 in the model to correspond to the 20′ of the shortened reed; the IGI of 4 is found to be 15′; "raising" (multiplying) 15′ to 20′ yields 5′ (represented by 1 in the model), which added to 20′ gives the original length 25′ NINDAN (VAT 7535, obv. 25f.; similarly rev., 22–24 and, apart from what looks like a copyists omission of a line, VAT 7532, rev. 7f.).

I have observed no uses of the term in mathematical texts outside of this specific kind of argument by single false position.

The conclusion to draw concerning the relation between *nasāḫum, tabālum, ḫarāṣum* and *šutbûm* is thus that *nasāḫum* is the fundamental term for identity-conserving subtraction. Other terms employed in daily life for processes where something is removed from a concrete totality may be used, firstly, in the formulation of "dressed problems" dealing with precisely such processes; but from there they might also creep into the description of mathematical procedures, in particular into places where the calculation evokes associations related to the everyday connotations of the term – either because of the real-world counterpart of the calculation or because of the structure of the model on which the calculation is based. This observation might hold for other parts of the mathematical vocabulary, too. We might say that the process by which a technical terminology is created was never brought to an end in Old (or, indeed, any) Babylonian mathematics.

The mathematical texts which come closest to revealing a technical vocabulary (for mathematical operations as well as for the real-world problems providing the dress) are the mathematical series texts.

In many respects the terminology used in these texts coincides with what we know from procedure texts. *nasāḫum* is used, while *tabālum, ḫarāṣum* and *šutbûm* are absent. LÁ (lal in MKT etc.), however, rises to unexpected prominence.

The term (in one case, its Akkadian equivalent *maṭûm*, "be(come) small(er)") only appears in two procedure texts. One of these is BM 13901. Here, No 10 tells that

> *mi-it-ḫar-tum a-na mi-it-ḫar-tim si-bi-a-tim im-ṭi*
> confrontation to confrontation, one seventh is smaller

while No 11 states that

> *mi-it-ḫar-tum* U.GÙ *mi-it-ḫar-tim si-bi-a-tim i-te-er*
> confrontation over confrontation, the seventh goes beyond.

The mathematical structure of the two problems is the same, apart from the fact that No 10 takes the fraction by which the two confrontations differ of the larger and No 11 of the smaller confrontation (in both cases, the larger confrontation is counted as the "first").

The reason for the different constructions is that Babylonian mathematics teachers had their favourite ways when formulating problems. One seventh is taken quite often (as are $^1/_{11}$, $^1/_{13}$, $^1/_{17}$, $^1/_{19}$, and $^1/_4$). One sixth and one eighth, on the other hand, are avoided as uninteresting. By comparing first the smaller to the larger, next the larger to the smaller, the author of the text has managed to use the favourite fraction $^1/_7$ both when the ratio is 7:6 and when it is 8:7.

maṭûm, precisely like *eli . . . watārum*, is thus a "subtraction by comparison", the only difference being the order of the operants. The same holds for the Sumerographic equivalents UGU . . . DIRIG and LÁ in the series texts. In YBC 4714 the reason for the change is precisely as in BM 13901: LÁ (or TUR, which is used synonymously in this tablet and nowhere else) is chosen when this choice makes it possible to refer to one of the favourite fractions while the use of UGU . . . DIRIG would preclude it; in neutral cases (e.g., when the difference is given in absolute value and not in relative terms), UGU . . . DIRIG is preferred (this is also the case in BM 13901).

The other occurrences of LÁ in the series texts have a slightly different explanation. They arise when, for some reason or other, the former of two magnitudes *A* and *B* which are compared comes out as the smaller. This can happen for a variety of reasons: *A* may be complex and *B* simple, as in YBC 4710, rev. ii 5–15;[12] one or both expressions may be submitted to systematic variation, and *A*

[12] Similarly also YBC 4673, rev. ii 16.

come out at times smaller, at times larger than B (as in YBC 4668, obv. iii 20–33, where $A > B$ in four cases and $A < B$ in two);[13] or a third entity C may be involved and consecutive lines deal with the amount by which C exceeds B and that by which A falls short of B (YBC 4708, rev. i 16, as corrected in MKT III 61). Since normal mathematical style would require the complex entity to be described first, and the spirit of systematization (as also the compact style of the series texts) would require that the order of entities be conserved in spite of variation of coefficients, all these cases can ultimately be traced back to considerations of style.

If the comparison between two systematically varied expressions A and B is translated into mathematical symbols and the order is made so as to reflect the text precisely, consecutive problems will be represented as in this example (MKT I 455, translating YBC 4668, obv. iii 20–24):

$$6{,}0(x-y)-(x+y)^2 = 18{,}20$$
$$2{,}30(x-y)-(x+y)^2 = -16{,}40$$

In Neugebauer's corresponding verbal translations (p. 440), the right-hand side of these equations become "18,20 geht es hinaus" and "16,40 ist es abgezogen".

When discussing expressions of this kind, Neugebauer would speak of "positivem Überschuß" and "negativem 'Abgezogenen'"; in the subject index he would refer to "Negative Größen" (MKT III 13, 83). What he meant by "negativity" was never more than this. It is thus with good reason that Neugebauer's *Exact Sciences in Antiquity* does not speak of "negative" but only of "subtractive numbers" or "subtractive writing of numbers" (1957: 236, 239).

Speaking of "negative numbers" in Babylonian mathematics is thus, firstly, a misreading of Neugebauer's much more restricted claim; secondly, it is unwarranted, unless one will proceed to claim that terms like "smaller", "below", and "before" also demonstrate knowledge of negative numbers, the "real" dimension to be measured being "larger", "above", and "after". All the more unwarranted, indeed, since the reasons to give the deficiency of A with regard to B instead of the excess the other way round turns out to depend on stylistic considerations or on the aspiration to make use of favourite fractions, and not on any attempt to investigate a particular mathematical conceptual structure or operation.

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13 So also YBC 4695, rev. i 11; YBC 4711, obv. i 13, 20, rev ii 36.

What we do have in a few texts are traces of an explicitly stated idea of "subtractive number" or "subtractive role of a number", in a wider sense than suggested by Neugebauer when he used the former expression.

One of these texts is BM 85200+VAT 6599 (the other procedure text in which LÁ occurs, *viz* in the passage to be quoted here). The statements of problems No 29 and 30 run, respectively,

> TÚL.SAG 1,40 UŠ IGI 7 *ša* UŠ U.GÙ SAG DIRIG *ù* 2 KÙŠ GAM-*ma* 3,20 [SAḪ]AR.ḪI⟨.A BA.ZI⟩
>
> A cellar. $1^{0}40'$ the length. The 7th part of that which the length over the width goes beyond, and 2 KÙŠ: the depth. $3^{0}20'$ of earth I have torn out

and

> TÚL.SAG 1,40 UŠ IGI 7 GÁL *ša* UŠ U.GÙ SAG DIRIG *ù* 1 KÙŠ BA.L[Á][14] GAM-*ma*
>
> A cellar. $1^{0}40'$ the length. The 7th part of that which the length over the width goes beyond, and 1 KÙŠ diminishing: the depth.

In No 29, the words *ù* 2 KÙŠ, "and 2 KÙŠ", and thus the word *ù*, are clearly additive/aggregative. If this understanding is transferred to the parallel formulation in No 30, the aggregation brings into play a *quantity to be subtracted.* Since the expression *ana* GAM 1 KÙŠ BA.LÁ (or 1 KÙŠ *imṭi,* according to BM 13901 No 10) was available, no apparent stylistic reasons enforce the particular construction used, and we may thus think of the two expressions as really reflecting the idea that the "normal role" of a number if aggregated is additive, but that the number may be marked (conceptually or materially) as possessing a subtractive role.

That the marking may indeed have been material is suggested by several passages in the text TMS XVI.[15] Line 8 quotes the statement in the phrase *aš-šum* 4-*at* SAG *na-sà-ḫu qa-bu-ku,* "since 'the fourth, to tear out', he has said to you". This unusual syllabic quotation of a logographically written statement makes it clear that the statement (line 1) [4-*at* SAG *i-na*] UŠ *ù* SAG ZI 45 should be read "the fourth of the width, from length and width to tear out" –

[14] For this crucial correction, see TMB 14.

[15] Cf. discussion of revised readings and of the mathematical structure of the text in Høyrup 1990: 299–306.

which again supports the reading of the damaged line 3 [50] *ù* 5 ZI ⌜GAR⌝ . . . as "50 and 5 to tear out pose".

"Posing" (*šakānum*/GAR) is a term which appears to possess several uses in the mathematical texts (cf. Høyrup 1990: 57f.). Common to these seems to be that a numerical value or other entity is taken note of in a calculational scheme or device or written/drawn materially. So, we must presume that the step to "pose" 50 (length+width) and 5 ($^1/_4$ of the width) implies that not only the number 5 but also its subtractive role is recorded.

A similar expression is encountered in line 23, . . . 45 *ta-⟨mar⟩ ki-ma* SAG GAR GAR ZI-*ma*, ". . . 45′ you see, as much as widths pose, pose to tear out" – i.e., the 45′ which result from the preceding calculation is to be recorded as the coefficient of the width together with the subtractive role of the resulting 45′·width.

Considered in isolation, each of these phrases from TMS XVI might be explained away as a stylistic slip or a dittography. Taken together, however, they appear to form a pattern, corroborating the assumption that Babylonian calculators would possess a notion of "numbers with a subtractive role". At the same time, however, they suggest that this role was bound up with material notations, perhaps through the way numbers were inserted into a calculational scheme or represented in a calculational device or similar representation. Nothing suggests that we are confronted with a specific *category* of numbers, say, with an incipient concept of negative numbers.

Instead we are led to the general conclusion that the Babylonian vocabulary for subtraction was somewhat fuzzy, employing a fairly large number of terms to describe only two different operations: Identity-conserving subtraction (*nasāḫum* etc.) and comparison (*eli . . . watārum*, LÁ); but that fixed techniques or calculational schemes were at hand which fully compensated for whatever lack of conceptual precision might follow from the blurred terminology.

Note added in proof: In a recent paper ("The Expressions of Zero and of Squaring in the Babylonian Mathematical Text VAT 7537", Historia Scientarum, 2nd series 1 (1991) 59–62), K. Muroi points to a case of subtraction by tearing-out where diminuend and subtrahend are equal. The result is stated as *ma-ṭi*, stative of *maṭûm*, to be interpreted as "it is missing". If we compare with the expression used in TMS V, section 11 (cf. above), where the subtrahend is told to be "as much as" (*kīma*) the diminuend in a subtraction by comparison, we notice that the ways to indicate a "zero outcome" (certainly not a resulting number zero) agree in semantics with the metaphorical origin of the respective subtractive operations.

Tablets Referred to

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